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# ON CLASSICAL MECHANICS CORRESPONDING TO THE DIRAC EQUATION –SUMMARY

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## IDEA AND RESULTS

We consider the free Dirac equation: For $\psi(t,q) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$,

(i) $$i\hbar\frac{\partial}{\partial t}\psi(t,q) = \mathbb{D}\psi(t,q), \quad \mathbb{D} = \sum_{k=1}^{3} c\alpha_k \frac{\hbar}{i}\frac{\partial}{\partial q_k} + mc^2\beta,$$

where we use the Dirac representation of matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We claim that there exists the "classical mechanics" corresponding to the Dirac equation, which is governed by the symbol obtained from the Dirac equation and that the Dirac equation is obtained by quantizing that symbol, that is, the Dirac equation is considered as the "Schrödinger picture" of the quantization of that symbol. This explains the usage of the term "Dirac particle" as reasonable comparing with that of the term " Schrödinger particle". In other word, we interpret the free Dirac equation as a quantum mechanical evolution equation, like Schrödinger equation for a single particle.

**Remark.** Rather dogmatically, we claim the meaning of classical mechanics as follows; The notion of classical mechanics consists of a configuration manifold $M$, its cotangent manifold $T^*M$, a Hamiltonian $H(q,p) \in C^\infty(T^*M : \mathbb{R})$ and the Poisson bracket $\{\cdot,\cdot\}$, by which $C^\infty(T^*M : \mathbb{R})$

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forms a Lie algebra. The object of classical mechanics is to investigate the structure of solutions of the equation of motion for each $H(q,p) \in C^\infty(T^*M : \mathbb{R})$. Whether the quantity derived by solving that equation is observable by suitable mechanical tools or not, is another problem.

To state our result more precisely, we proceed as follows (unfamiliar terminology will be given in Appendix):

(1) We identify a "spinor" $\psi(t,q) = {}^t(\psi_1(t,q), \psi_2(t,q), \psi_3(t,q), \psi_4(t,q)) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ with an even supersmooth function $u(t,x,\theta) = u_0(t,x) + u_1(t,x)\theta_1\theta_2 + u_2(t,x)\theta_2\theta_3 + u_3(t,x)\theta_3\theta_1 : \mathbb{R} \times \mathfrak{R}^{3|3} \to \mathfrak{C}_{ev}$. Here, $\mathfrak{R}^{3|3}$ is the superspace and $u_{j-1}(t,x)$ is the Grassmann continuation of $\psi_j(t,q)$ for $j = 1,2,3,4$. Then, matrices above are interpreted as differential operators acting on even supersmooth functions.

(2) Therefore, we may correspond the differential operator given by

$$
\begin{aligned}
\text{(ii)} \quad \mathcal{D}\Big(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\Big) = \Big(\theta_3 + \frac{\partial}{\partial \theta_3}\Big)\Big[c\Big(\theta_1 - \frac{\partial}{\partial \theta_1}\Big)\frac{\hbar}{i}\frac{\partial}{\partial x_1} + ic\Big(\theta_1 + \frac{\partial}{\partial \theta_1}\Big)\frac{\hbar}{i}\frac{\partial}{\partial x_2} \\
- c\Big(\theta_2 - \frac{\partial}{\partial \theta_2}\Big)\frac{\hbar}{i}\frac{\partial}{\partial x_3} + mc^2\Big(\theta_3 - \frac{\partial}{\partial \theta_3}\Big)\Big],
\end{aligned}
$$

which yields the superspace version of the free Dirac equation

$$
\text{(iii)} \qquad i\hbar\frac{\partial}{\partial t}u(t,x,\theta) = \mathcal{D}\Big(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\Big)u(t,x,\theta).
$$

Moreover, the "complete symbol" of (ii) is given by

$$
\begin{aligned}
\text{(iv)} \quad \mathcal{D}(x,\xi,\theta,\pi) = (\theta_3 + \pi_3)[c(\theta_1 - \pi_1)\xi_1 + ic(\theta_1 + \pi_1)\xi_2 - c(\theta_2 - \pi_2)\xi_3] \\
+ mc^2[1 + (\theta_3 + \pi_3)(\theta_3 - \pi_3)].
\end{aligned}
$$

*Remark.* In case of the free Dirac equation, $\mathcal{D}\big(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\big)$ and $\mathcal{D}(x,\xi,\theta,\pi)$ are independent of $x$.

(3) We consider the classical mechanics corresponding to $\mathcal{D}(x,\xi,\theta,\pi)$ given by

$$
\text{(v)} \qquad \begin{cases} \dfrac{d}{dt}x_j = \dfrac{\partial \mathcal{D}(x,\xi,\theta,\pi)}{\partial \xi_j}, & \dfrac{d}{dt}\xi_k = -\dfrac{\partial \mathcal{D}(x,\xi,\theta,\pi)}{\partial x_k}, \\[2mm] \dfrac{d}{dt}\theta_l = -\dfrac{\partial \mathcal{D}(x,\xi,\theta,\pi)}{\partial \pi_l}, & \dfrac{d}{dt}\pi_m = -\dfrac{\partial \mathcal{D}(x,\xi,\theta,\pi)}{\partial \theta_m}. \end{cases}
$$

In another word, using the graded Poisson bracket $\{\!\{\cdot,\cdot\}\!\}$ (see [3,4]), we have

$$
\frac{d}{dt}X(x(t),\xi(t),\theta(t),\pi(t)) = \{\!\{X,\mathcal{D}\}\!\}(x(t),\xi(t),\theta(t),\pi(t)) \quad \text{for any} \quad X \in \mathcal{C}_{SS}(\mathcal{T}^*\mathfrak{R}^{3|3} : \mathfrak{R}).
$$

(4) On the other hand, using Fourier transformation, we have readily that

$$\mathcal{D}\Big(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\Big) = \hat{\mathcal{D}}$$

where $\hat{\mathcal{D}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathcal{D}(x,\xi,\theta,\pi)$.

(5) Put

$$\begin{cases} \hat{x}_j u(x,\theta) = x_j u(x,\theta), \quad \hat{\xi}_k u(x,\theta) = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x_k} u(x,\theta) \quad \text{for} \quad j,k = 1,2,3, \\ \hat{\theta}_l u(x,\theta) = \theta_l u(x,\theta), \quad \hat{\pi}_m u(x,\theta) = \dfrac{\partial}{\partial \theta_m} u(x,\theta) \quad \text{for} \quad l,m = 1,2,3, \end{cases}$$

which yields therefore

$$[\hat{\xi}_k, \hat{x}_j]_- = \frac{\hbar}{i}\delta_{jk}, \quad [\hat{\pi}_m, \hat{\theta}_l]_+ = \delta_{lm} \quad \text{with} \quad [A,B]_\pm = AB \pm BA.$$

After proving that $\hat{\mathcal{D}}$ is self-adjoint on $\mathcal{L}^2_{SS}(\mathfrak{R}^{3|3} : \mathfrak{C})$ with domain $\mathcal{H}^1_{SS}(\mathfrak{R}^{3|3} : \mathfrak{C})$, we define

$$\begin{cases} \hat{x}^H_j(t) = e^{\frac{i}{\hbar}t\hat{\mathcal{D}}}\hat{x}_j e^{-\frac{i}{\hbar}t\hat{\mathcal{D}}}, \quad \hat{\xi}^H_k(t) = e^{\frac{i}{\hbar}t\hat{\mathcal{D}}}\hat{\xi}_k e^{-\frac{i}{\hbar}t\hat{\mathcal{D}}}, \\ \hat{\theta}^H_l(t) = e^{\frac{i}{\hbar}t\hat{\mathcal{D}}}\hat{\theta}_l e^{-\frac{i}{\hbar}t\hat{\mathcal{D}}}, \quad \hat{\pi}^H_m(t) = e^{\frac{i}{\hbar}t\hat{\mathcal{D}}}\hat{\pi}_m e^{-\frac{i}{\hbar}t\hat{\mathcal{D}}}, \end{cases}$$

which gives

$$\begin{cases} \dfrac{d}{dt}\hat{x}^H_j(t) = \dfrac{1}{i\hbar}[\hat{x}^H_j(t), \hat{\mathcal{D}}]_-, \quad \dfrac{d}{dt}\hat{\xi}^H_k(t) = \dfrac{1}{i\hbar}[\hat{\xi}^H_k(t), \hat{\mathcal{D}}]_-, \\ \dfrac{d}{dt}\hat{\theta}^H_l(t) = \dfrac{1}{i\hbar}[\hat{\theta}^H_l(t), \hat{\mathcal{D}}]_-, \quad \dfrac{d}{dt}\hat{\pi}^H_m(t) = \dfrac{1}{i\hbar}[\hat{\pi}^H_m(t), \hat{\mathcal{D}}]_-. \end{cases}$$

(6) We get also Ehrenfest type results partially: Put, for any non-zero $u(x,\theta) \in \mathcal{L}^2_{SS}(\mathfrak{R}^{3|3} : \mathfrak{C})$,

$$\langle \hat{X}(t) \rangle = \frac{(u, \hat{X}(t)u)}{(u,u)} \quad \text{for} \quad X = x_j \quad \text{etc},$$

then

$$\frac{d}{dt}\langle \hat{x}^H_j(t) \rangle = \frac{\partial \mathcal{D}}{\partial \xi_j}(\langle \hat{x}^H(t) \rangle, \langle \hat{\xi}^H(t) \rangle, \langle \hat{\theta}^H(t) \rangle, \langle \hat{\pi}^H(t) \rangle) = c\langle \hat{\alpha}^H_j(t) \rangle$$

which should be compaired with the first equation in (v),

$$\frac{d}{dt}x_j(t) = c\alpha_j(t) \quad \text{for} \quad j = 1,2,3,$$

where $\alpha_1(t) = \alpha_1(\theta(t), \pi(t)) = (\theta_3(t) + \pi_3(t))(\theta_1(t) - \pi_1(t))$, etc.

Concludingly, our Main assertions are as follows:

(1) There exists a Hamiltonian $\mathcal{D}(x,\xi,\theta,\pi)$ on $T^*\mathfrak{R}^{3|3} = \mathfrak{R}^{6|6}$ which defines the classical mechanics represented by

$$\frac{d}{dt}\varphi = \{\!\{\varphi, \mathcal{D}\}\!\} \quad \text{with} \quad \varphi(0,x,\xi,\theta,\pi) = \varphi_0(x,\xi,\theta,\pi).$$

(2) The pseudo-differential operator $\hat{\mathcal{D}}$ with symbol $\mathcal{D}(x,\xi,\theta,\pi)$ equals to $\mathcal{D}\Big(x,\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\partial}{\partial\theta}\Big)$, which is self-adjoint on $\mathcal{L}^2_{SS}(\mathfrak{R}^{3|3}:\mathfrak{C})$.

(3) The classical quantity $\varphi(t,x,\xi,\theta,\pi)$ evolving with has the quantum counter part, the operator valued functions(=observables) acting on $\mathcal{L}^2_{SS}(\mathfrak{R}^{3|3}:\mathfrak{C})$, which satisfies

$$i\hbar\frac{d}{dt}\hat{\varphi}=[\hat{\varphi},\hat{\mathcal{D}}]_- \quad\text{with}\quad \hat{\varphi}(0)=\hat{\varphi}_0.$$

Here $\hat{\varphi}(t)$ is the operator valued function of $t$ and $\hat{\varphi}_0$ is the pseudo-differential operator with symbol $\varphi_0(x,\xi,\theta,\pi)$. But, in general, $\hat{\varphi}(t)$ doesn't have the symbol which equals to $\varphi(t,x,\xi,\theta,\pi)$.

(4) There seems no analogue of Eherenfest type result for some operators such as the spin vector, though it has the classical counter part.

## Appendix. Fundamentals of superanalysis

For symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation

$$\sigma_j\sigma_k+\sigma_k\sigma_j=0,\quad j,k=1,2,\cdots,$$

we put

$$\mathfrak{C}=\{X=\sum_{I\in\mathcal{I}}X_I\sigma^I : X_I\in\mathbb{C}\}$$

where

$$\mathcal{I}=\{I=(i_k)\in\{0,1\}^{\mathbf{N}}:|I|=\sum_k i_k<\infty\},$$

$$\sigma^I=\sigma_1^{i_1}\sigma_2^{i_2}\cdots,\quad I=(i_1,i_2,\cdots),\sigma^{\tilde{0}}=1,\quad \tilde{0}=(0,0,\cdots)\in\mathcal{I}.$$

Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in $\mathfrak{C}$: For

$$X=\sum_{J\in\mathcal{J}}X_J\sigma^J,\quad Y=\sum_{K\in\mathcal{I}}Y_K\sigma^K,$$

we put

$$XY=\sum_{I\in\mathcal{I}}(XY)_I\sigma^I \quad\text{with}\quad (XY)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}X_JY_K.$$

Here, $\tau(I;J,K)$ is an integer defined by

$$\sigma^J\sigma^K=(-1)^{\tau(I;J,K)}\sigma^I,\quad I=J+K.$$

**Proposition.** *$\mathfrak{C}$ forms a $\infty$-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

*Remark.* (1) Degree in $\mathfrak{C}$ is defined by introducing subspaces

$$\mathfrak{C}_{[j]} = \{X = \sum_{I\in\mathcal{I},|I|=j} X_I\sigma^I\} \quad \text{for} \quad j=0,1,\cdots$$

which satisfy

$$\mathfrak{C} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[j]}, \quad \mathfrak{C}_{[j]}\cdot\mathfrak{C}_{[k]} \subset \mathfrak{C}_{[j+k]}.$$

(2) Define

$$\mathrm{proj}_I(X) = X_I \quad \text{for} \quad X = \sum_{I\in\mathcal{I}} X_I\sigma^I \in \mathfrak{C}.$$

The topology in $\mathfrak{C}$ is given by; $X \to 0$ in $\mathfrak{C}$ iff for any $I \in \mathcal{I}$, $\mathrm{proj}_I(X) \to 0$ in $\mathbb{C}$.

This topology is equivalent to the one introduced by the metric $\mathrm{dist}(X,Y) = \mathrm{dist}(X-Y)$ where $\mathrm{dist}(X)$ is defined by

$$\mathrm{dist}(X) = \sum_{I\in\mathcal{I}} \frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(X)|}{1+|\,\mathrm{proj}_I(X)|} \quad \text{with} \quad r(I) = 1+\frac{1}{2}\sum_{k=1}^{\infty}2^k i_k \quad \text{for} \quad I\in\mathcal{I}.$$

(3) We introduce parity in $\mathfrak{C}$ by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I\in\mathcal{I},|I|=ev} X_I\sigma^I, \\ 1 & \text{if } X = \sum_{I\in\mathcal{I},|I|=od} X_I\sigma^I, \\ \text{undefined} & \text{if otherwise.} \end{cases}$$

We put

$$\begin{cases} \mathfrak{C}_{ev} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[2j]} = \{X\in\mathfrak{C} : p(X)=0\}, \\ \mathfrak{C}_{od} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[2j+1]} = \{X\in\mathfrak{C} : p(X)=1\}, \\ \mathfrak{C} \cong \mathfrak{C}_{ev}\oplus\mathfrak{C}_{od} \cong \mathfrak{C}_{ev}\times\mathfrak{C}_{od}. \end{cases}$$

Analogous to $\mathfrak{C}$, we define

$$\begin{cases} \mathfrak{R} = \{X\in\mathfrak{C} : \pi_{\mathrm{B}}X \in \mathbb{R}\},\ \mathfrak{R}_{[j]} = \mathfrak{R}\cap\mathfrak{C}_{[j]}, \\ \mathfrak{R}_{ev} = \mathfrak{R}\cap\mathfrak{C}_{ev}, \quad \mathfrak{R}_{od} = \mathfrak{R}\cap\mathfrak{C}_{od} = \mathfrak{C}_{od}, \\ \mathfrak{R} \cong \mathfrak{R}_{ev}\oplus\mathfrak{R}_{od} \cong \mathfrak{R}_{ev}\times\mathfrak{R}_{od}. \end{cases}$$

We introduced the body (projection) map $\pi_{\mathrm{B}}$ by

$$\pi_{\mathrm{B}}X = \mathrm{proj}_{\bar{0}}(X) = X_{\bar{0}} = X_{\mathrm{B}} \quad \text{for any} \quad X \in \mathfrak{C}.$$

We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n} = \mathfrak{R}^m_{ev} \times \mathfrak{R}^n_{od}.$$

The metric between $X, Y \in \mathfrak{R}^{m|n}$ is defined by,

$$\mathrm{dist}_{m|n}(X,Y) = \mathrm{dist}_{m|n}(X-Y)$$

with

$$\mathrm{dist}_{m|n}(X) = \sum_{j=1}^{m}\left(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(x_j)|}{1+|\,\mathrm{proj}_I(x_j)|}\right) + \sum_{k=1}^{n}\left(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(\theta_k)|}{1+|\,\mathrm{proj}_I(\theta_k)|}\right).$$

We use the following notations:

$$X = (X_A)_{A=1}^{m+n} = (x,\theta) \in \mathfrak{R}^{m|n} \quad \text{with}$$
$$x = (X_A)_{A=1}^{m} = (x_j)_{j=1}^{m} \in \mathfrak{R}^{m|0}, \quad \theta = (X_A)_{A=m+1}^{m+n} = (\theta_k)_{k=1}^{n} \in \mathfrak{R}^{0|n}.$$

We generalize the body map $\pi_{\mathrm{B}}$ from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\mathbb{R}^m$ by putting,

$$X = (x,\theta) \in \mathfrak{R}^{m|n} \longrightarrow \pi_{\mathrm{B}}X = X_{\mathrm{B}} = (x_{\mathrm{B}},0) \cong x_{\mathrm{B}} = \pi_{\mathrm{B}}x = (\pi_{\mathrm{B}}x_1,\cdots,\pi_{\mathrm{B}}x_m) \in \mathbb{R}^m.$$

We call $x_j \in \mathfrak{R}_{ev}$ even (alias bosonic) variable and $\theta_k \in \mathfrak{R}_{od}$ odd (alias ferminioc) variable, respectively.

For

$$\mathfrak{a} = (\alpha, a),\ \alpha = (\alpha_1,\cdots,\alpha_m) \in \mathbf{N}^m,\ a = (a_1,\cdots,a_n) \in \{0,1\}^n,$$
$$|\alpha| = \sum_{j=1}^{m}\alpha_j,\ \ |a| = \sum_{k=1}^{n}a_k,\ \ |\mathfrak{a}| = |\alpha| + |a|,$$

we put

$$\partial_X^{\mathfrak{a}} = \partial_x^{\alpha}\partial_\theta^{a} \quad \text{with} \quad \partial_x^{\alpha} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m}, \quad \partial_\theta^{a} = \partial_{\theta_1}^{a_1}\cdots\partial_{\theta_n}^{a_n}.$$

*Example.* $\partial_{\theta_2}\theta_1\theta_2\theta_3 = -\theta_1\theta_3, \quad \partial_{\theta_1}\partial_{\theta_3}\theta_1\theta_2\theta_3 = \theta_2, \quad$ *etc.*

For $u_a(q) \in C^\infty(\mathbb{R}^m;\mathbb{C})$, we put, for $x = x_{\mathrm{B}} + x_{\mathrm{S}}$,

$$u_a(x) = \sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\partial_q^{\alpha}u_a(x_{\mathrm{B}})x_{\mathrm{S}}^{\alpha}$$

which is called the Grassmann continuation of $u_a(q)$ and define a function $u \in \mathcal{C}_{SS}(\mathfrak{R}^{m|n})$ by

$$u(X) = u(x,\theta) = \sum_{|a|\le n} u_a(x)\theta^a.$$

**Scalar products and norms:**

$$(u,v) = \int_{\mathfrak{R}^{m|n}} dx d\theta\, d\bar\theta e^{\langle\bar\theta|\theta\rangle}\overline{u(x,\theta)}v(x,\theta) = \sum_{|a|\le n}\int_{\mathfrak{R}^{m|0}} dx\, \overline{u_a(x)}v_a(x),$$

$$((u,v))_k = \sum_{|a|\le k}(\partial_X^a u, \partial_X^a v) = \sum_{|\alpha|+|a|\le k}(\partial_x^\alpha u_a, \partial_x^\alpha v_a),$$

$$(((u,v)))_k = \sum_{|a|+l\le k}((1+|X_{\mathrm{B}}|^2)^{l/2}\partial_X^a u, (1+|X_{\mathrm{B}}|^2)^{l/2}\partial_X^a v)$$

with

$$||u||^2 = (u,u), \quad ||u||_k^2 = ((u,u))_k, \quad |||u|||_k^2 = (((u,u)))_k.$$

**Fourier transformations:**

$$(F_e v)(\xi) = (2\pi\hbar)^{-m/2}\int_{\mathfrak{R}^{m|0}} dx\, e^{-i\hbar^{-1}\langle\xi|x\rangle}v(x), \quad (\bar F_e w)(x) = (2\pi\hbar)^{-m/2}\int_{\mathfrak{R}^{m|0}} d\xi\, e^{i\hbar^{-1}\langle\xi|x\rangle}w(\xi),$$

$$(F_o v)(\pi) = \jmath_n\int_{\mathfrak{R}^{0|n}} d\theta\, e^{-\langle\pi|\theta\rangle}v(\theta), \quad (\bar F_o w)(\theta) = \jmath_n\int_{\mathfrak{R}^{0|n}} d\pi\, e^{\langle\pi|\theta\rangle}w(\pi)$$

where

$$\langle\eta|y\rangle = \sum_{j=1}^m \eta_j y_j, \quad \langle\rho|\omega\rangle = \sum_{k=1}^n \rho_k\omega_k, \quad \jmath_n = e^{\frac{\pi i}{4}n(n-1)}.$$

We put

$$(\mathcal{F}u)(\xi,\pi) = c_{m,n}\int_{\mathfrak{R}^{m|n}} dX\, e^{-i\langle\Xi|X\rangle}u(X) = \sum_a [(F_e u_a)(\xi)][(F_o\theta^a)(\pi)],$$

$$(\bar{\mathcal{F}}v)(x,\theta) = c_{m,n}\int_{\mathfrak{R}^{m|n}} d\Xi\, e^{i\langle\Xi|X\rangle}v(\Xi) = \sum_a [(\bar F_e v_a)(x)][(\bar F_o\pi^a)(\theta)]$$

where

$$\langle\Xi|X\rangle = \hbar^{-1}\langle\xi|x\rangle - i\langle\pi|\theta\rangle \in \mathfrak{R}_{ev}, \quad c_{m,n} = (2\pi\hbar)^{-m/2}\jmath_n.$$

*Remark.* Though the differential calculus on Fréchet spaces has some difficulties in general, such a calculus on Fréchet-Grassmann algebra holds safely in our case. For example, the implicit and inverse function theorems, and the chain rule of differentials are established as similar as the standard case.

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